JOURNAL OF APPROXIMATION THEORY 7, 334–351 (1973)

Polynomial Approximation on Tetrahedrons in the Finite Element Method

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Communicated by Oved Shisha

Received December 24, 1970

The main aim of this paper is to derive an interpolation theorem (Theorem 1) which implies both a construction of once continuously differentiable functions which are piecewise polynomial in a domain divided into tetrahedrons (Corollary 1 and Theorem 2) and convergence theorems of the finite element method for solving three-dimensional elliptic boundary value problems of the fourth order (Theorems 3 and 4).

1. INTRODUCTION

It is well known [1] that the simplest polynomial in one variable generating piecewise polynomial functions which are m-times continuously differentiable is a polynomial of degree $2m + 1$. This polynomial is uniquely determined by the function values and all derivatives up to the m-th order inclusive at the end-points of a segment. There exist general interpolation theorems (see, e.g., [2]) from which the convergence of the finite element method follows.

It is also known (see [3], [4]) that, in the general case, the simplest polynomial $p(x, y)$ on a triangle, generating piecewise polynomial functions which are m-times continuously differentiable in a triangulated domain is a polynomial of degree $4m + 1$. The conditions uniquely determining it are of such a form that considering $p(x, y)$ along the side P_iP_j of the triangle, i.e., setting

$$x = x_i + (x_j - x_i)s, \qquad y = y_i + (y_j - y_i)s, \qquad 0 \leqslant s \leqslant 1,$$

we obtain a polynomial $p(s)$ of degree $4m + 1$ determined in such a way that it generates, as a polynomial in one variable s, $2m$-times continuously differentiable functions.

Extrapolating this fact to the case of three variables it may be expected that the simplest polynomial $p(x, y, z)$ on the tetrahedron generating piecewise polynomial functions which are m-times continuously differentiable should be of such a degree and determined by such conditions that considering it on the triangular face $P_iP_jP_k$ of the tetrahedron we obtain a polynomial $p(s, t)$ which generates, as a polynomial in two variables s, t, $2m$-times continuously differentiable functions. Thus the degree of such a polynomial should be $8m + 1$.

The case $m = 0$ is trivial. As to $m = 1$ and $m = 2$ the expectation was confirmed to be true (see [5]). Attempts to do this in the general case have not yet been successful.

The aim of this paper is to derive an interpolation theorem for the polynomial of the ninth degree (Theorem 1 and Corollary 1) and using it to prove the convergence theorems of the finite element method for solving three-dimensional variational problems of the second order which are equivalent to elliptic boundary value problems of the fourth order. The method of the proof of Theorem 1 is a modification of the method which was developed in the case of two variables in [6] and then generalized in [3].

2. Notation

A given closed tetrahedron will be denoted by $\overline{U}$, its interior by U. The vertices and the center of gravity of $\overline{U}$ will be denoted by P_i ($i = 1,..., 4$) and P_0, respectively. The centers of gravity of the triangular faces $P_2P_3P_4$, $P_1P_3P_4$, $P_1P_2P_4$, and $P_1P_2P_3$ are denoted by Q_1, Q_2, Q_3, and Q_4, respectively. The symbols $Q_{jk}^{(1,s)},..., Q_{jk}^{(s,s)}$ denote the points dividing the segment $\langle P_jP_k\rangle$ into $s + 1$ equal parts.

The symbols s_{jk}, t_{jk} mean two arbitrary but fixed directions such that the directions P_jP_k, s_{jk}, t_{jk} are perpendicular to one another.

The symbol n_i denotes the normal to the triangular face the center of gravity of which is the point Q_i. We orientate n_i according to the right-hand screw rule with respect to the increasing indices $j < k < l$ of the vertices P_j, P_k, P_l of the face. The symbols s_i and t_i mean two arbitrary but fixed directions such that n_i, s_i, t_i are perpendicular to one another.

Let P_j, P_k be two vertices of the triangular face the center of gravity of which is the point Q_i. The symbol ν_{ijk} denotes the direction perpendicular to the directions n_i and P_jP_k.

Let f be a function of the variables x, y, z and $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, $\alpha_3 \geqslant 0$ three arbitrary integers. Setting

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \qquad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

the operator D^α is defined by

$$D^\alpha f = \partial^{|\alpha|} f / \partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}.$$

Similarly, if g is a function of the variables ξ, η, ζ then

$$D^\alpha g = \partial^{|\alpha|} g / \partial \xi^{\alpha_1} \partial \eta^{\alpha_2} \partial \zeta^{\alpha_3}.$$

Let $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$ be two arbitrary integers. Setting

$$\beta = (\beta_1, \beta_2), \qquad |\beta| = \beta_1 + \beta_2,$$

the operators D_i^β and D_{jk}^β are defined by

$$D_i^\beta f = \partial^{|\beta|} f / \partial s_i^{\beta_1} \partial t_i^{\beta_2}, \qquad D_{jk}^\beta f = \partial^{|\beta|} f / \partial s_{jk}^{\beta_1} \partial t_{jk}^{\beta_2}$$

where $\partial f / \partial s_i$, $\partial f / \partial t_i$, $\partial f / \partial s_{jk}$, and $\partial f / \partial t_{jk}$ denote the derivatives of the function f in the directions s_i, t_i, s_{jk}, and t_{jk}, respectively. Further, if φ is a function of two variables ξ, η we define

$$D^\beta \varphi = \partial^{|\beta|} \varphi / \partial \xi^{\beta_1} \partial \eta^{\beta_2}.$$

The symbols $\partial f / \partial n_i$ and $\partial f / \partial \nu_{ijk}$ denote the derivatives of the function f in the directions n_i and ν_{ijk}, respectively.

3. Interpolation Theorem

THEOREM 1. *Let the function $w(x, y, z)$ be continuous on a closed tetrahedron $\overline{U}$ and have bounded derivatives of the tenth order in the interior U of $\overline{U}$:*

$$|D^\alpha w(x, y, z)| \leqslant M_{10}, \qquad |\alpha| = 10, \qquad (x, y, z) \in U. \tag{1}$$

Let

$$D^\alpha w(P_i) = 0, \qquad |\alpha| \leqslant 4; \tag{2}$$

$$D_{jk}^\beta w(Q_{jk}^{(r,s)}) = 0, \qquad |\beta| = s, \quad r = 1,..., s; \quad s = 1, 2; \tag{3}$$

$$w(Q_i) = 0; \tag{4}$$

$$D_i^\beta(\partial w(Q_i)/\partial n_i) = 0, \qquad |\beta| \leqslant 2; \tag{5}$$

$$D^\alpha w(P_0) = 0, \qquad |\alpha| \leqslant 1; \tag{6}$$

where $i = 1,..., 4, j = 1, 2, 3, k = 2, 3, 4$ $(j < k)$. Then it holds on $\overline{U}$

$$|D^\alpha w(x, y, z)| \leqslant \frac{K}{q^{|\alpha|} V^{|\alpha|+1}} M_{10} h^{10-|\alpha|}, \qquad |\alpha| \leqslant 8 \tag{7}$$

where h is the length of the largest edge of the tetrahedron $\bar{U}$ and K is a constant independent on $\bar{U}$ and on $w(x, y, z)$. The constant V is defined by

$$V = \min(V_1, \dots, V_4), \tag{8}$$

V_i being the volume of the unit parallelepiped having edges parallel to the edges of $\bar{U}$ which intersect at the vertex P_i. The quotient q is defined by

$$q = \max_{i=1,\dots,4} \min(a_i/h, b_i/h, c_i/h), \tag{9}$$

a_i, b_i and c_i being the lengths of the edges having the vertex P_i as a common point.

The interpolation character of Theorem 1, which will be proved in Section 4, follows from the following.

COROLLARY 1. *A polynomial of the ninth degree*

$$p(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + \cdots + a_{220} z^9 \tag{10}$$

is uniquely determined by the conditions (2)–(6) *where*

$$w(x, y, z) = p(x, y, z) - f(x, y, z), \tag{11}$$

$f(x, y, z)$ being a function four-times continuously differentiable on the tetrahedron $\bar{U}$. Further, if the function $f(x, y, z)$ has bounded derivatives of the tenth order in the interior U of $\bar{U}$,

$$|D^\alpha f(x, y, z)| \leqslant M_{10}, \qquad |\alpha| = 10, \qquad (x, y, z) \in U,$$

then the difference (11) *satisfies the inequality* (7).

Proof. The number of the conditions (2)–(6) is equal to 220. (The numbers of the conditions (2), (3), (4), (5), and (6) are equal to 140, 48, 4, 24, and 4, respectively.) If $w(x, y, z)$ is of the form (11) then the conditions (2)–(6) form a system of 220 linear equations for the 220 unknown coefficients $a_1, \dots, a_{220}$. It is sufficient to prove that the determinant of this system is different from zero.

Let us assume that the function $w(x, y, z) = p(x, y, z)$ satisfies the conditions (2)–(6). As, according to Eq. (10),

$$D^\alpha p(x, y, z) \equiv 0, \qquad |\alpha| = 10$$

it follows from Theorem 1 that $p(x, y, z) \equiv 0$. The inverse implication is trivial. Corollary 1 is proved.

Remark. The estimates for derivatives in (7) depend on the geometry of the tetrahedron. It is natural to ask whether this dependence is essential or a mere consequence of the used method of proving. An analogous situation exists in case of interpolation polynomials on the triangle (see [3, 4, 6] or Lemma 3). Though this problem has not yet been generally solved the following is worth mentioning.

1. The quantity V is a three-dimensional analogy of $\sin \omega$, ω being the smallest angle of a given triangle, because $\sin \omega$ is the measure of the unit rhombus having sides parallel to the sides making the angle ω.

2. In the two-dimensional case the estimates for derivatives depend just on $\sin \omega$ because $1/q < 2$. In the three-dimensional case the quantity $1/q$ is unbounded. (It suffices to consider the tetrahedron with vertices $P_1(0, 0, 0)$, $P_2(1, 0, 0)$, $P_3(0, \epsilon, 0)$, and $P_4(1, 0, \epsilon)$.)

3. Ciarlet and Wagschal [7] derived by means of multipoint Taylor formulas the following estimates for the first derivatives in case of interpolation polynomials of the first, second, and third degree on n-dimensional simplexes:

$$|(\partial\varphi/\partial x_i) - (\partial p_r/\partial x_i)| \leqslant C_r M_{r+1}(h^{r+1}/h') \qquad (i = 1, \ldots, n;\, r = 1, 2, 3),$$

$p_r(x_1, \ldots, x_n)$ being the interpolation polynomial of the r-th degree of the function $\varphi(x_1, \ldots, x_n)$. C_r is a constant independent on the simplex and on the function $\varphi(x_1, \ldots, x_n)$, M_{r+1} is the bound of the derivatives of the order $r + 1$ of the function $\varphi(x_1, \ldots, x_n)$, h is the length of the largest edge of the simplex and h' the diameter of the inscribed sphere of the simplex.

4. Let us consider the tetrahedron with vertices $P_1(-h/2, -k_1h, 0)$, $P_2(h/2, -k_1h, 0)$, $P_3(0, k_2h, 0)$ and $P_4(0, 0, z_0)$ where h, k_1, k_2, z_0 are positive numbers satisfying

$$k_1 + k_2 \leqslant 3^{1/2}/2, \qquad z_0 \leqslant h \min(((3/4) - k_1^2)^{1/2}, (1 - k_2^2)^{1/2}).$$

Under these conditions h is the length of the largest edge of the tetrahedron $P_1P_2P_3P_4$.

The second derivatives of the function

$$f(x, y, z) = h^2 z + 4x^2 + k_2^{-1}(k_1 + k_2)^{-1}\, y(y + k_1 h) - h^2$$

are bounded, $M_2 = \max(8, 2k_2^{-1}(k_1 + k_2)^{-1})$, and the interpolation polynomial of the first degree of the function f reads:

$$p_1(x, y, z) = h^2(1 - z_0^{-1})z.$$

It holds

$$|(\partial f/\partial z) - (\partial p_1/\partial z)| = h^2/z_0 .$$

If $z_0 \to 0+$ then both V and h' tend to zero and

$$|(\partial f/\partial z) - (\partial p_1/\partial z)| \to \infty.$$

The example introduced proves in case of interpolation polynomials of the first degree that the estimates for derivatives are dependent on the geometry of the tetrahedron.

4. Some Lemmas and Proof of Theorem 1

LEMMA 1. *Let $g(s)$ be a function of a real parameter $s \in [0, l]$, continuous on $[0, l]$ and having a bounded derivative of the order $n+1$ in $(0, l)$,*

$$|g^{(n+1)}(s)| \leqslant K_{n+1}, \qquad s \in (0, l).$$

Let

$$s_0 = 0 < s_1 < s_2 < \cdots < s_r = l$$

$$|g^{(k)}(s_i)| \leqslant \eta_i^{(k)} \qquad (k = 0,..., \alpha_i - 1; i = 0,..., r)$$

where $\eta_i^{(k)}$ are constants and α_i given integers satisfying

$$\alpha_0 + \alpha_1 + \cdots + \alpha_r = n + 1, \qquad \alpha_i \geqslant 1.$$

Further, let

$$\eta = \max_{i=0,\ldots,r} (\max_{k=0,\ldots,\alpha_i-1} l^k \eta_i^{(k)}).$$

Then

$$|g^{(j)}(s)| \leqslant C_{2j+1} l^{-j} \eta + C_{2j+2} K_{n+1} l^{n+1-j}, \qquad s \in (0, l)$$

where $j = 0, 1,..., n-1$. $C_1, C_2,..., C_{2n}$ are constants independent on the function $g(s)$ and on the interval $[0, l]$.

LEMMA 2. *Let*

$$|\partial^n f(P)/\partial s_1^{i_1} \cdots \partial s_m^{i_m}| \leqslant M, \qquad i_1 + \cdots + i_m = n,$$

P being a point in the space x, y, z and $s_1,..., s_m$ $(2 \leqslant m \leqslant 3)$ arbitrary directions perpendicular to one another. Then

$$|\partial^n f(P)/\partial l_1 \partial l_2 \cdots \partial l_n| \leqslant m^{n/2} M$$

where $l_1, l_2,..., l_n$ are arbitrary directions dependent on the directions $s_1,..., s_m$.

Lemma 1 is proved in [3, Theorem 2]. Lemma 2 can be obtained by means of Schwarz's inequality. The following lemma is a slight modification of [3, Theorem 4], and of [8, Theorem 13].

LEMMA 3. *Let the function* $u(\xi, \eta)$ *have bounded derivatives of the order* $n + 1$ *on the closed triangle* $\overline{T}$ ($n = 8$ or 9):

$$| D^{\beta}u(\xi, \eta)| \leqslant N_{n+1}, \qquad | \beta | = n + 1, \qquad (\xi, \eta) \in \overline{T}.$$

Let

$$| D^{\beta}u(R_i)| \leqslant \epsilon, \qquad | D^{\gamma}u(R_0)| \leqslant \epsilon, \qquad | \partial^j u(S_r^{(k)})/\partial \nu^j | \leqslant \epsilon$$

where $\partial u/\partial \nu$ *is the normal derivative,* R_i ($i = 1, 2, 3$) *the vertices of* $\overline{T}$, R_0 *the center of gravity of* $\overline{T}$, $S_r^{(k)}$ ($r = 1,..., 3k$) *the points dividing the sides of* $\overline{T}$ *into* $k + 1$ *equal parts and where the indices* β, γ, j, k *are determined in the case* $n = 8$ *by*

$$| \beta | \leqslant 3, \qquad | \gamma | \leqslant 2, \qquad j = k - 1, \qquad k = 1, 2$$

and in the case $n = 9$ *by*

$$| \beta | \leqslant 4, \qquad | \gamma | = 0, \qquad j = k = 1, 2.$$

Then it holds on $\overline{T}$

$$| D^{\beta}u(\xi, \eta)| \leqslant \vartheta_n + (K_n/(\sin \omega)^{|\beta|}) N_{n+1}c^{n+1-|\beta|}, \qquad | \beta | \leqslant n - 1$$

where $\vartheta_n \to 0$ *if* $\epsilon \to 0+$; c *is the length of the largest side of* $\overline{T}$, ω *the smallest angle of* $\overline{T}$ *and* K_n *a constant independent on* $\overline{T}$ *and on* $u(\xi, \eta)$.

In what follows we shall use Sobolev's spaces $W_2^{(k)}(\Omega)$ and $\tilde{W}_2^{(k)}(\Omega)$ Ω being a connected bounded domain in the space (x, y, z). $W_2^{(k)}(\Omega)$ is the space of functions having generalized derivatives up to the order k inclusive which belong to the space $L_2(\Omega)$. The norm in $W_2^{(k)}(\Omega)$ is defined by

$$\| w \|^2_{W_2^{(k)}(\Omega)} = \sum_{|\alpha| \leqslant k} \| D^{\alpha}w \|^2_{L_2(\Omega)} .$$

The space $\tilde{W}_2^{(k)}(\Omega)$ consists of functions which together with all generalized derivatives of the order k belong to $L_2(\Omega)$. The norm is given by

$$\| w \|^2_{\tilde{W}_2^{(k)}(\Omega)} = \| w \|^2_{L_2(\Omega)} + \sum_{|\alpha| = k} \| D^{\alpha}w \|^2_{L_2(\Omega)} .$$

In the following considerations we shall often need Sobolev's lemma in the following special form (see [9]).

LEMMA 4 (Sobolev). *Let Ω be a domain starlike with respect to a sphere. Let $0 \leqslant m \leqslant k-2$ and $w \in \tilde{W}_2^{(k)}(\Omega)$. Then $w \in C^{(m)}(\bar{\Omega})$ and*

$$\max_{(x,y,z)\in\bar{\Omega},|\alpha|\leqslant m} | D^\alpha w(x, y, z)| \leqslant C\| w \|_{\tilde{W}_2^{(k)}(\Omega)}$$

where the constant C does not depend on $w(x, y, z)$.

Two parts of the proof of Theorem 1 will be used in the proof of Theorem 2. We formulate them, therefore, in Lemmas 5 and 6.

LEMMA 5. *Let the function $w(x, y, z)$ be continuous on a closed tetrahedron $\bar{U}$ and the inequality* (1) *hold. Let $\bar{T}_{\rho\sigma\tau}$ be the triangular face of $\bar{U}$ with vertices P_ρ, P_σ, P_τ ($\rho < \sigma < \tau$) and Q_λ the center of gravity of $\bar{T}_{\rho\sigma\tau}$. If Eq.* (2) *holds for $i = \rho, \sigma, \tau$, Eq.* (3) *holds for $j = \rho, \sigma$, $k = \sigma, \tau$ ($j < k$) and Eqs.* (4), (5) *hold for $i = \lambda$, then*

$$| D^\alpha w(P)| \leqslant (K_1/V^{|\alpha|})\, M_{10} h^{10-|\alpha|}, \qquad |\alpha| \leqslant 1, \qquad P \in \bar{T}_{\rho\sigma\tau}. \tag{12}$$

The meaning of V and h is the same as in Theorem 1 *and K_1 is a constant independent on $\bar{U}$ and on $w(x, y, z)$.*

Proof. It follows from the assumptions of Lemma 5 that the function $w(x, y, z)$ belongs to $\tilde{W}^{(10)}(U)$. Thus, according to Lemma 4, the function $w(x, y, z)$ is eight-times continuously differentiable on $\bar{U}$.

Let us construct a tetrahedron $\bar{U}'$ with vertices P_ρ', P_σ', P_τ', P_λ' lying inside $\bar{U}$. Let the faces $\bar{U}'$ be parallel to the faces of $\bar{U}$ and lie in a distance δ. Choosing δ sufficiently small it holds with respect to the assumptions of Lemma 5:

$$| D^\alpha w(P_i')| \leqslant \epsilon/3^{|\alpha|/2}, \qquad |\alpha| \leqslant 4,\ i = \rho, \sigma, \tau; \tag{13}$$

$$| D^\beta_{jk} w(\bar{Q}^{(r,s)}_{jk})| \leqslant \epsilon/2^{|\beta|/2}, \qquad |\beta| = s;\ r = 1,..., s;\ s = 1, 2;\ j = \rho, \sigma;\ k = \sigma, \tau \quad (j < k) \tag{14}$$

$$| w(Q_\lambda')| \leqslant \epsilon \tag{15}$$

$$| D_\lambda^\beta(\partial w(Q_\lambda')/\partial n_\lambda)| \leqslant \epsilon, \qquad |\beta| \leqslant 2 \tag{16}$$

where Q_λ' is the center of gravity of the triangle $P_\rho' P_\sigma' P_\tau'$ and $\bar{Q}^{(1,s)}_{jk},..., \bar{Q}^{(s,s)}_{jk}$ are the points dividing the segment $\langle P_j' P_k' \rangle$ into $s + 1$ equal parts. Using Lemma 2 we obtain from (13) and (14):

$$| \partial^{|\alpha|} w(P_i')/\partial s_\lambda^{\alpha_1}\, \partial t_\lambda^{\alpha_2}\, \partial n_\lambda^{\alpha_3} | \leqslant \epsilon, \qquad |\alpha| \leqslant 4 \tag{17}$$

$$| \partial^s w(\bar{Q}^{(r,s)}_{jk})/\partial v^s_{\lambda jk} | \leqslant \epsilon, \qquad r = 1,..., s;\quad s = 1, 2. \tag{18}$$

Let ξ, η, ζ be a Cartesian coordinate system the (ξ, η)-plane of which is identical with the plane determined by the points P_ρ', P_σ', P_τ'. Let the directions of the axes ξ, η, and ζ be parallel to the directions s_λ, t_λ, and n_λ, respectively. Let

$$\begin{aligned} x &= x(\xi, \eta, \zeta) \equiv \bar{x} + a_{11}\xi + a_{12}\eta + a_{13}\zeta \\ y &= y(\xi, \eta, \zeta) \equiv \bar{y} + a_{21}\xi + a_{22}\eta + a_{23}\zeta \\ z &= z(\xi, \eta, \zeta) \equiv \bar{z} + a_{31}\xi + a_{32}\eta + a_{33}\zeta \end{aligned} \tag{19}$$

be the transformation between the systems x, y, z and ξ, η, ζ, $(\bar{x}, \bar{y}, \bar{z})$ being the coordinates of the origin of the system ξ, η, ζ in the system x, y, z. Let us define the function

$$\tilde{w}(\xi, \eta, \zeta) = w(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)). \tag{20}$$

Then, according to (1), (15)–(20), and Lemma 2, it is easy to see that the functions

$$\varphi(\xi, \eta) = \tilde{w}(\xi, \eta, 0) \tag{21}$$

and

$$\psi(\xi, \eta) = \partial\tilde{w}(\xi, \eta, 0)/\partial\zeta \tag{22}$$

satisfy the conditions of Lemma 3 with $N_{10} = 3^5 M_{10}$ and $N_9 = 3^5 M_{10}$, respectively. Hence, according to Lemma 3 and (21), (22), it holds for $(\xi, \eta, \zeta) \in \bar{T}'_{\rho\sigma\tau}$

$$| D^\alpha \tilde{w}(\xi, \eta, \zeta)| \leqslant \vartheta + (AM_{10}/(\sin \omega)^{|\alpha|})\, \bar{c}^{10-|\alpha|}, \qquad |\alpha| \leqslant 1 \tag{23}$$

where $\vartheta \to 0$ if $\epsilon \to 0+$. $\bar{T}'_{\rho\sigma\tau}$ is the triangle with vertices P_i' $(i = \rho, \sigma, \tau)$, $\bar{c}$ is the length of the largest side of $\bar{T}'_{\rho\sigma\tau}$, ω is the smallest angle of $\bar{T}'_{\rho\sigma\tau}$ and A is a constant independent on $\bar{T}'_{\rho\sigma\tau}$ and on $\tilde{w}(\xi, \eta, \zeta)$. Further, it holds

$$\sin \omega \geqslant V, \qquad \lim_{\epsilon \to 0+} \bar{c} = c \leqslant h.$$

Thus, returning to the variables x, y, z and letting $\epsilon \to 0+$, we obtain by means of (23) (with respect to orthogonality of the matrix of the transformation (19)) the inequality (12).

LEMMA 6. *Let the function $w(x, y, z)$ be continuous on a closed tetrahedron $\bar{U}$ and the inequality* (1) *hold. Let P_ρ, P_σ be two vertices of the tetrahedron $\bar{U}$. If Eq.* (2) *holds for $i = \rho, \sigma$ and Eq.* (3) *holds for $j = \rho$, $k = \sigma$ then*

$$| D^\alpha w(P)| \leqslant K_2 M_{10} h^{10-|\alpha|}, \qquad |\alpha| \leqslant 2, \qquad P \in \langle P_\rho P_\sigma \rangle$$

where h is the length of the largest edge of $\overline{U}$ and K_2 is a constant independent on $\overline{U}$ and on $w(x, y, z)$.

Making use of Lemma 1 we can prove Lemma 6 in a similar way as Lemma 5.

Proof of Theorem 1. Let us choose the notation of the vertices $P_i(x_i, y_i, z_i)$ $(i = 1,..., 4)$ in such a way that

$$q = \min(a_1/h, b_1/h, c_1/h), \tag{24}$$

a_1, b_1 and c_1 being the lengths of the segments $\langle P_1P_2\rangle$, $\langle P_1P_3\rangle$, and $\langle P_1P_4\rangle$, respectively. Let (ρ_1, ρ_2, ρ_3), $(\sigma_1, \sigma_2, \sigma_3)$, and (τ_1, τ_2, τ_3) be the unit vectors which are parallel to the directions P_1P_2, P_1P_3, and P_1P_4, respectively. Then the transformation

$$\begin{aligned} x &= x(\xi, \eta, \zeta) \equiv x_1 + a_1\rho_1\xi + b_1\sigma_1\eta + c_1\tau_1\zeta \\ y &= y(\xi, \eta, \zeta) \equiv y_1 + a_1\rho_2\xi + b_1\sigma_2\eta + c_1\tau_2\zeta \\ z &= z(\xi, \eta, \zeta) \equiv z_1 + a_1\rho_3\xi + b_1\sigma_3\eta + c_1\tau_3\zeta \end{aligned} \tag{25}$$

maps one-to-one the tetrahedron $\overline{U}$ on the tetrahedron $\overline{U}_0$ which lies in the Cartesian coordinate system ξ, η, ζ and has the vertices $R_1(0, 0, 0)$, $R_2(1, 0, 0)$, $R_3(0, 1, 0)$, and $R_4(0, 0, 1)$. We shall distinguish between two cases: $M_{10} > 0$ and $M_{10} = 0$.

In the case $M_{10} > 0$ let us introduce the function

$$v(\xi, \eta, \zeta) = M_{10}^{-1}h^{-10}w(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)). \tag{26}$$

It holds with respect to (25) and (26)

$$\frac{\partial^{|\alpha|}v(\xi, \eta, \zeta)}{\partial\xi^{\alpha_1}\,\partial\eta^{\alpha_2}\,\partial\zeta^{\alpha_3}} = \frac{a_1^{\alpha_1}b_1^{\alpha_2}c_1^{\alpha_3}}{M_{10}h^{10}}\,\frac{\partial^{|\alpha|}w(x, y, z)}{\partial\rho^{\alpha_1}\,\partial\sigma^{\alpha_2}\,\partial\tau^{\alpha_3}},$$

$\partial w/\partial\rho$, $\partial w/\partial\sigma$, and $\partial w/\partial\tau$ being the derivatives in the directions P_1P_2, P_1P_3, and P_1P_4, respectively. Hence, according to (1)–(6) and Lemmas 2, 5, and 6,

$$| D^\alpha v(\xi, \eta, \zeta)| \leqslant 3^5, \qquad | \alpha | = 10, \quad (\xi, \eta, \zeta) \in U_0; \tag{27}$$

$$D^\alpha v(R_i) = 0, \qquad | \alpha | \leqslant 4, \quad i = 1,..., 4; \tag{28}$$

$$D^\alpha v(R_0) = 0, \qquad | \alpha | \leqslant 1; \tag{29}$$

$$| D^\alpha v(R)| \leqslant 3^{|\alpha|/2}K_1V^{-|\alpha|}, \qquad | \alpha | \leqslant 1, \quad R \in \overline{U}_0\backslash U_0; \tag{30}$$

$$| D^\alpha v(R)| \leqslant 3^{|\alpha|/2}K_2, \qquad | \alpha | \leqslant 2, \quad R \in \langle R_iR_j\rangle \quad (i \neq j;\, i, j = 1,..., 4); \tag{31}$$

where U_0 is the interior of $\overline{U}_0$ and R_0 the center of gravity of $\overline{U}_0$. The function $v(\xi, \eta, \zeta)$, being continuous on $\overline{U}_0$ and having bounded derivatives of the tenth order in U_0, belongs to $\tilde{W}_2^{(10)}(U_0)$. Thus, according to Lemma 4, $v(\xi, \eta, \zeta)$ is eight-times continuously differentiable on $\overline{U}_0$.

Let us assume that we succeeded in proving the inequalities

$$| D^\alpha v(R_1)| \leqslant C_1 V^{-1}, \qquad | \alpha | = 5, 6; \tag{32}$$

$$| D^\alpha v(R_2)| \leqslant C_2 V^{-1}, \qquad | \alpha | = 5, 6; \tag{33}$$

$$| D^\alpha v(R_4)| \leqslant C_3 V^{-1}, \qquad | \alpha | = 5. \tag{34}$$

(Here and in the following text the symbols $C_1, \ldots, C_{21}$ denote absolute constants, i.e. constants independent on the function $w(x, y, z)$ and on the tetrahedron $\overline{U}$.) Let us consider the function

$$g_1(s) = v \,|_{\langle R_4 S_4 \rangle},$$

S_4 being the center of gravity of the triangular face $R_1R_2R_3$. As the lengths of the segments $\langle R_4S_4\rangle$ and $\langle R_4R_0\rangle$ are equal to $11^{1/2}/3$ and $11^{1/2}/4$, respectively, it holds, according to (27)–(30), (34), and Lemma 2,

$$| g_1^{(10)}(s)| \leqslant 3^{10}, \qquad s \in (0, 11^{1/2}/3)$$

$$| g_1^{(j)}(0)| \leqslant 3^{j/2} C_3 V^{-1}, \qquad g_1^{(k)}(11^{1/2}/4) = 0$$

$$| g_1^{(k)}(11^{1/2}/3)| \leqslant 3^k K_1 V^{-k} \qquad (j = 0, \ldots, 5;\ k = 0, 1).$$

Using Lemma 1 we obtain

$$| g_1(s)| \leqslant C_4 V^{-1} \max(3^{5/2} C_3, 3K_1) + 11^5 C_5, \qquad s \in [0, 11^{1/2}/3].$$

As $V^{-1} > 1$ we can write, setting $C_6 = C_4 \max(3^{5/2} C_3, 3K_1) + 11^5 C_5$,

$$| v(R)| \leqslant C_6 V^{-1}, \qquad R \in \langle R_4 S_4 \rangle. \tag{35}$$

Now, let R be an arbitrary point of the interior of the triangle $R_2R_4S_{13}$, S_{13} being the midpoint of the segment $\langle R_1R_3\rangle$. Let R_5 be the crossing point of the triangular face $R_1R_3R_4$ and the line determined by the points R_2, R. Let us consider the function

$$g_2(s) = v \,|_{\langle R_2 R_5 \rangle}.$$

It holds, according to (27) and Lemma 2,

$$| g_2^{(10)}(s) | \leqslant 3^{10}, \; s \in (0, l),$$

l being the length of the segment $\langle R_2R_5\rangle$. Denoting by l_1 the distance between

R_2 and the crossing point of the segments $\langle R_2R_5\rangle$ and $\langle R_4S_4\rangle$ we can write with respect to (28), (30), (33), and (35):

$$| g_2^{(j)}(0)| \leqslant 3^{j/2}C_2V^{-1}, \qquad | g_2(l_1)| \leqslant C_6V^{-1}$$

$$| g_2^{(k)}(l)| \leqslant 3^kK_1V^{-k} \qquad (j = 0,\ldots, 6;\ k = 0, 1).$$

As $l < 2^{1/2}$ we obtain by means of Lemma 1

$$| v(R)| \leqslant C_7V^{-1}, \qquad R \in T, \tag{36}$$

T being the interior of the triangle $R_2R_4S_{13}$.

At the end, let R be an arbitrary point of U_0. Denoting by R' the crossing point of the line R_1R and the triangular face $R_2R_3R_4$ and considering the function $g_3(s) = v|_{\langle R_1R'\rangle}$ we can prove by means of (27), (28), (30), (32), (36), and Lemma 1, similarly as in the case of the function $g_2(s)$, that it holds

$$| v(R)| \leqslant C_8V^{-1}, \qquad R \in U_0 . \tag{37}$$

The estimates (27), (30), and (37) imply

$$\| v \|_{\tilde{W}_2^{(10)}(U_0)} \leqslant C_9V^{-1}. \tag{38}$$

Making use of Lemma 4 we get from (38)

$$| D^\alpha v(\xi, \eta, \zeta)| \leqslant C_{10}V^{-1}, \qquad | \alpha | \leqslant 8, \qquad (\xi, \eta, \zeta) \in \overline{U}_0 . \tag{39}$$

As $| J | = a_1b_1c_1V_1 \geqslant a_1b_1c_1V$, J being the Jacobian of the transformation (25), we get from (25)

$$\begin{array}{lll} | \partial\xi/\partial x |, & | \partial\xi/\partial y |, & | \partial\xi/\partial z | \leqslant a_1^{-1}V^{-1} \\ | \partial\eta/\partial x |, & | \partial\eta/\partial y |, & | \partial\eta/\partial z | \leqslant b_1^{-1}V^{-1} \\ | \partial\zeta/\partial x |, & | \partial\zeta/\partial y |, & | \partial\zeta/\partial z | \leqslant c_1^{-1}V^{-1}. \end{array} \tag{40}$$

Expressing the function $w(x, y, z)$ in the form

$$w(x, y, z) = M_{10}h^{10}v(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$

we obtain by means of (24), (39), and (40) the estimation (7).

To finish the proof it remains to prove the inequalities (32)–(34). Let $\overline{U}_0'$ be a tetrahedron lying inside $\overline{U}_0$ and having faces parallel to the faces of $\overline{U}_0$ in a distance δ. Choosing δ sufficiently small it holds with respect to (30) and (31)

$$| D^\alpha v(R)| \leqslant \epsilon + 3^{|\alpha|/2}K_1V^{-|\alpha|}, \qquad | \alpha | \leqslant 1, \quad R \in \overline{U}_0'\backslash U_0' \tag{41}$$

$$| D^\alpha v(R)| \leqslant \epsilon + 3^{|\alpha|/2}K_2 , \qquad | \alpha | \leqslant 2, \quad R \in \langle R_i'R_j'\rangle, \tag{42}$$

U_0' being the interior of $\overline{U}_0'$ and R_i' $(i = 1,..., 4)$ the vertices of $\overline{U}_0'$. Let us consider the functions

$$g_{\alpha_2\alpha_3}(\xi) = \partial^{\alpha_2+\alpha_3}v/\partial\eta^{\alpha_2}\,\partial\zeta^{\alpha_3}\,|_{\langle R_1'R_2'\rangle}\,, \qquad \alpha_2 + \alpha_3 \leqslant 2.$$

It holds, according to (27) and (42),

$$|\, g_{\alpha_2\alpha_3}^{(10-\alpha_2-\alpha_3)}(\xi)| \leqslant 3^5, \qquad |\, g_{\alpha_2\alpha_3}(\xi)| \leqslant \epsilon + 3K_2\,, \qquad \xi \in [0, l],$$

l being the length of the segment $\langle R_1'R_2'\rangle$. Using Lemma 1 and letting $\epsilon \to 0+$ we obtain

$$|\, D^\alpha v(\xi, 0, 0)| \leqslant C_{11}\,, \qquad \xi \in [0, 1], \qquad |\, \alpha\,| = 5, 6; \quad \alpha_2 + \alpha_3 \leqslant 2. \tag{43}$$

Considering the functions $\partial^{\alpha_1+\alpha_3}v/\partial\xi^{\alpha_1}\,\partial\zeta^{\alpha_3}(\alpha_1 + \alpha_3 \leqslant 2)$ and $\partial^{\alpha_1+\alpha_2}v/\partial\xi^{\alpha_1}\,\partial\eta^{\alpha_2}$ $(\alpha_1 + \alpha_2 \leqslant 2)$ on the segments $\langle R_1'R_3'\rangle$ and $\langle R_1'R_4'\rangle$, respectively, we prove

$$|\, D^\alpha v(0, \eta, 0)| \leqslant C_{12}\,, \qquad \eta \in [0, 1], \qquad |\, \alpha\,| = 5, 6; \quad \alpha_1 + \alpha_3 \leqslant 2; \tag{44}$$

$$|\, D^\alpha v(0, 0, \zeta)| \leqslant C_{13}\,, \qquad \zeta \in [0, 1], \qquad |\, \alpha\,| = 5, 6; \quad \alpha_1 + \alpha_2 \leqslant 2. \tag{45}$$

In the case $|\, \alpha\,| = 5$ it remains to estimate the derivatives

$$D^{(2,2,1)}v(R_1), \qquad D^{(2,1,2)}v(R_1), \qquad D^{(1,2,2)}v(R_1). \tag{46}$$

Let us consider the function

$$g_4(s) = \partial v/\partial\zeta\,|_{\langle R_1'S_{23}'\rangle}\,,$$

S_{23}' being the midpoint of the segment $\langle R_2'R_3'\rangle$. It holds, according to (27), (41), and Lemma 2,

$$|\, g_4^{(9)}(s)| \leqslant 3^{10}, \qquad |\, g_4(s)| \leqslant \epsilon + 3K_1V^{-1}, \qquad s \in [0, l'],$$

l' being the length of the segment $\langle R_1'S_{23}'\rangle$. Using Lemma 1 and letting $\epsilon \to 0+$ we obtain

$$|\, \partial^5 v(R_1)/\partial s^4\,\partial\zeta\,| \leqslant C_{14}V^{-1} \tag{47}$$

where $\partial v/\partial s$ is the derivative in the direction $(2^{1/2}/2, 2^{1/2}/2, 0)$. The estimates (43), (44), and (47) imply

$$|\, D^{(2,2,1)}v(R_1)| \leqslant C_{15}V^{-1}$$

with $C_{15} = 5(C_{11} + C_{12})/6 + 2C_{14}/3$. The last two derivatives (46) can be estimated similarly by considering the functions $\partial v/\partial\eta$ and $\partial v/\partial\xi$ on the segments $\langle R_1'S_{24}'\rangle$ and $\langle R_1'S_{34}'\rangle$, respectively.

To prove (32) it remains to estimate the derivatives

$$D^{(3,3,0)}v(R_1), \qquad D^{(3,0,3)}v(R_1), \qquad D^{(0,3,3)}v(R_1); \tag{48}$$
$$D^{(3,2,1)}v(R_1), \qquad D^{(2,3,1)}v(R_1); \tag{49}$$
$$D^{(3,1,2)}v(R_1), \qquad D^{(2,1,3)}v(R_1); \tag{50}$$
$$D^{(1,3,2)}v(R_1), \qquad D^{(1,2,3)}v(R_1); \tag{51}$$
$$D^{(2,2,2)}v(R_1). \tag{52}$$

In the case of (48) we can manage it by considering the functions $\partial v/\partial \eta$, $\partial v/\partial \xi$, and $\partial v/\partial \zeta$ on the segments $\langle R_1'S_{23}'\rangle$, $\langle R_1'S_{24}'\rangle$, and $\langle R_1'S_{34}'\rangle$, respectively.

The derivatives (49) will be estimated simultaneously. Let us consider the functions

$$g_5(s) = \partial v/\partial \zeta \,|_{\langle R_1' Q_{23}'\rangle}\,, \qquad g_6(t) = \partial v/\partial \zeta \,|_{\langle R_1' Q_{23}''\rangle}\,,$$

Q_{23}' and Q_{23}'' being the points which divide the segment $\langle R_2'R_3'\rangle$ into thirds. The inequalities (27) and (41) imply by means of Lemma 1

$$|\,\partial^6 v(R_1)/\partial s^5\,\partial\zeta\,| \leqslant C_{16}V^{-1}, \qquad |\,\partial^6 v(R_1)/\partial t^5\,\partial\zeta\,| \leqslant C_{17}V^{-1} \tag{53}$$

where $\partial v/\partial s$ and $\partial v/\partial t$ denote the derivatives in the directions $(2(5^{1/2}/5), 5^{1/2}/5, 0)$ and $(5^{1/2}/5, 2(5^{1/2}/5), 0)$, respectively. It follows from (43), (44), and (53)

$$|\,D^{(3,2,1)}v(R_1)| \leqslant C_{18}V^{-1}, \qquad |\,D^{(2,3,1)}v(R_1)| \leqslant C_{19}V^{-1}.$$

The derivatives (50) and (51) can be estimated similarly.

Having estimated all derivatives $D^\alpha v(R_1)(|\,\alpha\,| = 6)$ except for (52) we can derive

$$|\,D^{(2,2,2)}v(R_1)| \leqslant C_{20}V^{-1}$$

by considering the function $g_7(s) = v\,|_{\langle R_1S_1\rangle}$, S_1 being the center of gravity of the triangular face $R_2R_3R_4$.

To derive (33) let us map the tetrahedron $\overline{U}_0$ by the transformation

$$\xi = 1 - \kappa - \lambda - \chi, \qquad \eta = \lambda, \qquad \zeta = \chi \tag{54}$$

on the tetrahedron $\overline{U}_1$ lying in the Cartesian coordinate system κ, λ, χ and having the vertices $A_1(0, 0, 0)$, $A_2(1, 0, 0)$, $A_3(0, 1, 0)$, and $A_4(0, 0, 1)$. Defining the function

$$u(\kappa, \lambda, \chi) = v(1 - \kappa - \lambda - \chi, \lambda, \chi)$$

and repeating the preceding considerations we obtain

$$|\,D^\alpha u(A_1)| \leqslant C_{21}V^{-1}, \qquad |\,\alpha\,| = 5, 6. \tag{55}$$

As the point R_2 is mapped by (54) on the point A_1 the estimation (55) implies (33). The estimation (34) can be obtained similarly. Theorem 1 is proved in the case $M_{10} > 0$.

If $M_{10} = 0$ then the inequality

$$| D^\alpha w(x, y, z)| \leqslant \Delta, \qquad | \alpha | = 10, \qquad (x, y, z) \in U$$

holds for arbitrary $\Delta \geqslant 0$. Repeating the preceding proof with $M_{10} = \Delta > 0$, where Δ is arbitrarily small, and letting $\Delta \to 0+$ we complete the proof of Theorem 1.

5. Applications

Let Ω be a bounded simply or multiply connected domain in E_3 with the boundary Γ consisting of a finite number of polyhedrons Γ_i $(i = 0,..., s)$; Γ_1 ,..., Γ_s lie inside of Γ_0 and do not intersect. Let $\mathfrak{M}$ be a set of a finite number of closed tetrahedrons having the following properties: (1) the union of all tetrahedrons is $\bar{\Omega}$; (2) two arbitrary tetrahedrons are either disjoint or have a common vertex or a common edge or a common face.

Let N_t, N_v, and N_f be the total numbers of the tetrahedrons, of the vertices and of the triangular faces in the division $\mathfrak{M}$, respectively. The tetrahedrons of $\mathfrak{M}$ will be denoted by $\bar{U}_i$ $(i = 1,..., N_t)$, the vertices by P_i $(i = 1,..., N_v)$ and the triangular faces by $\bar{T}_i$ $(i = 1,..., N_f)$. The symbol Q_i denotes now the center of gravity of the triangle $\bar{T}_i$. The normal n_i to the face $\bar{T}_i$ is oriented according to the rule introduced in Section 2. The meaning of the symbols $Q_{jk}^{(1,s)},..., Q_{jk}^{(s,s)}$ is the same as in Section 2. The center of gravity of the tetrahedron $\bar{U}_i$ is denoted by $P_0^{(i)}$. Similarly as in Section 2 to each edge P_jP_k there are prescribed two directions s_{jk}, t_{jk} and to each normal n_i two directions s_i , t_i .

Let there be prescribed at each point P_i thirty-five values $D^\alpha f(P_i)$ ($| \alpha | \leqslant 4$), at each point $Q_{jk}^{(1,1)}$ two values $D_{jk}^\beta f(Q_{jk}^{(1,1)})$($| \beta | = 1$), at each point $Q_{jk}^{(r,2)}$ three values $D_{jk}^\beta f(Q_{jk}^{(r,2)})$ ($| \beta | = 2$), at each point $P_0^{(i)}$ four values $D^\alpha f(P_0^{(i)})$ ($| \alpha | \leqslant 1$) and at each point Q_i one value $f(Q_i)$ and six values $D_i^\beta \, \partial f(Q_i)/\partial n_i$ ($| \beta | \leqslant 2$). Then on each tetrahedron $\bar{U}_i$ there is uniquely determined a polynomial of the ninth degree $p_i(x, y, z)$ and the following theorem holds.

Theorem 2. *The function*

$$g(x, y, z) = p_i(x, y, z), \qquad (x, y, z) \in \bar{U}_i \quad (i = 1,..., N_t) \tag{56}$$

is once continuously differentiable on the domain $\bar{\Omega}$.

Theorem 2 was proved in [5]. However, another proof of Theorem 2 follows immediately from Lemma 5. Let the tetrahedrons $\bar{U}_{i_1}$ and $\bar{U}_{i_2}$ have the triangle $\bar{T}_\lambda$ as a common face. Let P_ρ, P_σ, P_τ be the vertices of $\bar{T}_\lambda$. Then the polynomial $p(x, y, z) = p_{i_1}(x, y, z) - p_{i_2}(x, y, z)$ satisfies all assumptions of Lemma 5 with $M_{10} = 0$. Thus, according to (12),

$$D^\alpha p_{i_1}(x, y, z) = D^\alpha p_{i_2}(x, y, z), \qquad |\alpha| \leqslant 1, \qquad (x, y, z) \in \bar{T}_\lambda .$$

Moreover, making use of Lemma 6 we can prove in the same way that the function (56) is twice continuously differentiable on the edges $\langle P_j P_k \rangle$ in the division $\mathfrak{M}$.

Let us denote the set of all functions of the type (56) by $G(\mathfrak{M})$. The set $G(\mathfrak{M})$ is a finite-dimensional space with

$$\dim G(\mathfrak{M}) = 35N_v + 7N_f + 4N_t + 8N_e$$

where the integers N_v, N_f, and N_t are defined above and N_e is the total number of the edges $\langle P_j P_k \rangle$ in the division $\mathfrak{M}$.

It is clear that $G(\mathfrak{M}) \subset W_2^{(2)}(\Omega)$. Thus we can use the functions of the type (56) as trial functions in the finite element procedure for solving three-dimensional boundary value problems of elliptic equations of the fourth order. We restrict ourselves to the variational formulation of the problem.

Let $H \subset W_2^{(2)}(\Omega)$ be a real Hilbert space with the norm induced by $W_2^{(2)}(\Omega)$. Let $a(v, w)$ be a real bilinear form continuous on $H \times H$, i.e., a mapping $(v, w) \to a(v, w)$ from $H \times H$ into the field of real numbers which is linear in both v and w and bounded:

$$| a(v, w)| \leqslant M \| v \|_{W_2^{(2)}(\Omega)} \| w \|_{W_2^{(2)}(\Omega)} , \qquad \forall v, w \in H \tag{57}$$

where M is a constant independent on v, w. Further, let the form $a(v, w)$ be symmetric,

$$a(v, w) = a(w, v), \ \forall v,w \in H, \tag{58}$$

and H-elliptic, i.e.,

$$a(v, v) \geqslant \kappa \| v \|^2_{W_2^{(2)}(\Omega)} , \qquad \forall v \in H \tag{59}$$

where $\kappa > 0$ is a constant independent on v. Finally, let $L(v)$ be a linear functional continuous on H. Then (see [10]) there exists just one $u \in H$ such that

$$a(u, v) = L(v), \qquad \forall v \in H. \tag{60}$$

It is well known that u satisfies Eq. (60) if and only if u minimizes sharply on H the functional

$$F(v) = (1/2)a(v, v) - L(v). \tag{61}$$

The space H is determined by the stable homogeneous boundary conditions of the boundary value problem to which the given variational problem corresponds. In our case of tetrahedral elements we must restrict our considerations to such cases when the part Γ' of Γ on which the stable boundary conditions are prescribed can be covered by a finite number of triangles. In this case we can choose the division $\mathfrak{M}$ in such a way that Γ' is a union of some triangular faces $\bar{T}_i$.

The approximate solution of the given variational problem is then defined as the function which minimizes the functional (61) on the space $G(\mathfrak{M}) \cap H$. ($G(\mathfrak{M}) \cap H$ is the space of all functions of $G(\mathfrak{M})$ satisfying the stable boundary conditions in the classical sense.) It follows immediately from (59) that there exists just one function of this property.

Now, let $\{\mathfrak{M}_h\}$ be a set of divisions of $\bar{\Omega}$ into closed tetrahedrons with the following properties:

$$h \to 0, \; q_h \geqslant q_0 > 0, \; V_h \geqslant V_0 > 0, \tag{62}$$

h being the length of the largest edge in $\mathfrak{M}_h$, q_h the smallest quantity (9) in $\mathfrak{M}_h$ and V_h the smallest quantity (8) in $\mathfrak{M}_h$. Let $H_h = G(\mathfrak{M}_h) \cap H$ and u_h be the approximate solution of the given variational problem on H_h. The following two convergence theorems hold.

THEOREM 3. *Under the assumptions* (57)–(59) *and* (62) *it holds*

$$\lim_{h\to 0} \| u_h - u \|_{W_2^{(2)}(\Omega)} = 0, \tag{63}$$

u being the exact solution of the given variational problem.

The proof of Theorem 3 goes in the same lines as the proof of the convergence theorem introduced in [11]; instead of [6, Theorem 3] we use Theorem 1. Further, Theorem 1 allows us to state a sufficient condition for the maximum rate of convergence.

THEOREM 4. *Let the conditions* (57)–(59) *and* (62) *be satisfied and the exact solution $u(x, y, z)$ have bounded derivatives of the tenth order in Ω,*

$$| D^\alpha u(x, y, z)| \leqslant M_{10}, \qquad | \alpha | = 10, \qquad (x, y, z) \in \Omega. \tag{64}$$

Then

$$\| u_h - u \|_{W_2^{(2)}(\Omega)} \leqslant CM_{10}h^8 \tag{65}$$

where the constant C does not depend on the division $\mathfrak{M}$ and on the exact solution $u(x, y, z)$.

Proof. According to [12, p. 365], it holds

$$\| u_h - u \|_{W_2^{(2)}(\Omega)} \leqslant M^{1/2}\kappa^{-1/2} \| u - v \|_{W_2^{(2)}(\Omega)} , \qquad \forall v \in H_h .$$

Let φ be the function from H_h having the same values at the points P_i, $Q_{jk}^{(r,s)}$, $P_0^{(i)}$, Q_i as the exact solution u. Making use of Corollary 1 we can state

$$\| u - \varphi \|_{W_2^{(2)}(\Omega)} \leqslant C' M_{10} h^8$$

where the constant C' depends on q_0, V_0 and mes Ω only. As $\varphi \in H_h$ the last two inequalities imply the estimate (65). Theorem 4 is proved.

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